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On the Cesàro Means of Conjugate Jacobi Series*

Zhongkai Li

Department of Mathematics, Capital Normal University, Beijing 100037, China

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This paper deals with the Cesàro means of conjugate Jacobi series introduced by Muckenhoupt and Stein and Li. The exact estimates of the norms of the conjugate (C, δ) kernel for $0 \leqslant \delta \leqslant \alpha + \frac{1}{2}$ are obtained. It is proved that when $\delta > \alpha + \frac{1}{2}$, the [illegible] the critical index under the criterion of Lebesgue type by use of the equiconvergence theorem.

1. INTRODUCTION

Let $\{R_n^{(\alpha, \beta)}(\cos\theta)\}$ be the sequence of the Jacobi polynomials of order (α, β), normalized so that $R_n^{(\alpha, \beta)}(1) = 1$, which is orthogonal over $(0, \pi)$ with respect to the measure

$$d\mu(\theta) = d\mu_{\alpha, \beta}(\theta) = 2^{\alpha+\beta+1} \sin^{2\alpha+1}\theta/2 \cos^{2\beta+1}\theta/2 d\theta.$$

Denote by $L = L_{\alpha\beta}$ the class of functions integrable with respect to $d\mu_{\alpha\beta}(\theta)$ on $(0, \pi)$.

For a function $f \in L$, its Jacobi expansion is

$$f(\theta) \sim \sum_{n=0}^{\infty} \hat{f}(n)\, \omega_n^{(\alpha, \beta)} R_n^{(\alpha, \beta)}(\cos\theta), \tag{1}$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\varphi)\, R_n^{(\alpha, \beta)}(\cos\varphi)\, d\mu_{\alpha\beta}(\varphi)$$

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are the Fourier coefficients and

$$\omega_n^{(\alpha,\beta)} = \left\{ \int_0^\pi [R_n^{(\alpha,\beta)}(\cos\varphi)]^2 \, d\mu_{\alpha\beta}(\varphi) \right\}^{-1} \sim n^{2\alpha+1}.$$

To a Jacobi series of the form (1), its conjugate series is defined by

$$\sum_{n=1}^{\infty} \frac{(2\alpha+2)\,\hat{f}(n)}{n+\alpha+\beta+1}\, \omega_{n-1}^{(\alpha+1,\beta+1)} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta. \tag{2}$$

This is introduced by Muckenhoupt and Stein [7] when $\alpha=\beta$, and by Li [4] for general α and β. It is also noted in Bavinck [1, Section 6.2] and in an unpublished note of Gasper.

On the other hand, the (Jacobi) conjugate function $\tilde{f}$ of a function f is defined in Li [4] by

$$\tilde{f}(\theta) = \lim_{\varepsilon\to 0} \tilde{f}_\varepsilon(\theta), \tag{3}$$

where

$$\tilde{f}_\varepsilon(\theta) = \int_\varepsilon^\pi \tilde{T}_\varphi f(\theta)\cdot G(\varphi)\, d\mu_{\alpha\beta}(\varphi),$$

is the generalized truncated Hilbert transform, where for $\alpha>\beta>-\frac{1}{2}$,

$$\begin{aligned}
\tilde{T}_\varphi f(\theta) &= \int_0^\pi \int_0^1 f(\psi)\, d\tilde{m}_{\alpha\beta}(t,\zeta) \\
&\sim \sum_{k=1}^{\infty} \hat{f}(k)\, \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \\
&\quad \times R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi)\sin\theta\sin\varphi, \qquad (4)\\
\cos\psi &= 2(\cos\theta/2\cos\varphi/2)^2 + 2(t\sin\theta/2\sin\varphi/2)^2 \\
&\quad + t\sin\theta\sin\varphi\cos\zeta - 1,\\
d\tilde{m}_{\alpha\beta}(t,\zeta) &= c_{\alpha\beta}(1-t^2)^{\alpha-\beta-1}\, t^{2\beta+2} \sin^{2\beta}\zeta\cos\zeta\, dt\, d\zeta,\\
c_{\alpha\beta} &= \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac{1}{2})}
\end{aligned}$$

and

$$G(\varphi) = (2\alpha+2)\int_0^1 s^{\alpha+\beta+1} P^{(\alpha+1,\beta+1)}(s,\varphi)\sin\varphi\, ds, \tag{5}$$

and

$$P^{(\alpha,\beta)}(s,\varphi)=\sum_{n=0}^{\infty} s^n\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\varphi)$$

is the Poisson kernel. If $\alpha=\beta>-\frac{1}{2}$ or $\alpha>\beta=-\frac{1}{2}$, we can obtain an appropriate form for $\tilde{T}_\varphi$ by a limiting way.

It has been shown in Li [4, Theorem 2; 5, Corollary 4] that for any $f\in L_{\alpha\beta}$, the limit in (3) exists for almost every $\theta\in(0,\pi)$ and the Abel means of the conjugate Jacobi series (2) converges to $\tilde{f}(\theta)$ almost everywhere, and if $\tilde{f}\in L_{\alpha\beta}$, then $\tilde{f}$ has the expansion (2), namely,

$$\tilde{f}(\theta)\sim\sum_{n=1}^{\infty}\frac{(2\alpha+2)\,\hat{f}(n)}{n+\alpha+\beta+1}\,\omega_{n-1}^{(\alpha+1,\beta+1)}R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta. \tag{6}$$

In this paper, we study the Cesàro means of the conjugate (Jacobi) series (6) which is defined as

$$\tilde{S}_n^\delta(f;\theta)=\frac{1}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^\delta\frac{(2\alpha+2)\,\hat{f}(\nu+1)}{\nu+\alpha+\beta+2}\,\omega_\nu^{(\alpha+1,\beta+1)}R_\nu^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta,$$

where $A_\rho^\sigma=\Gamma(\rho+\sigma+1)/\Gamma(\sigma+1)\,\Gamma(\rho+1)$. The kernel form of $\tilde{S}_n^\delta$ is

$$\tilde{S}_n^\delta(f;\theta)=\int_0^\pi \tilde{T}_\varphi f(\theta)\cdot\tilde{K}_n^\delta(\varphi)\,d\mu_{\alpha\beta}(\varphi),$$

where

$$\tilde{K}_n^\delta(\varphi)=\frac{2\alpha+2}{A_n^\delta}\sum_{\nu=0}^{n}A_{n-\nu}^\delta\frac{\omega_\nu^{(\alpha+1,\beta+1)}}{\nu+\alpha+\beta+2}R_\nu^{(\alpha+1,\beta+1)}(\cos\varphi)\sin\varphi$$

is the conjugate Cesàro kernel. This follows from (4).

Just as the case of trigonometric series, one may expect to prove for conjugate Jacobi series the parallel results to those about the Cesàro means of Jacobi series in [6]. In Section 2, the estimates for the conjugate Cesàro kernel $\tilde{K}_n^\delta$ for $\delta\geqslant 0$ are obtained. Section 3 is devoted to the evaluation of the norm of $\tilde{K}_n^\delta$ for $\delta\leqslant\alpha+\frac{1}{2}$. In Section 4, the pointwise convergence theorem of $\tilde{S}_n^\delta(f;\theta)$ when $\delta>\alpha+\frac{1}{2}$ is established and the equiconvergence theorem of the Cesàro means $\tilde{S}_n^{\alpha+1/2}(f;\theta)$ at the critical index is proved and then applied to establish the convergence theorem of Lebesgue type.

It has been indicated in Li [4] that when $\alpha=\beta=-\frac{1}{2}$, $\tilde{T}_\varphi f(\theta)$ and $G(\varphi)$ reduce to $(f(\theta-\varphi)-f(\theta+\varphi))/2$ and $(1/\pi)$ ctg $\varphi/2$, respectively, so that the previous definition of $\tilde{f}$ coincides with the conjugate function in classical case. In addition, when $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, $\tilde{T}_\varphi f(\theta)$ has some properties of continuity so that it can be used as a measure of smoothness of f.

PROPOSITION 1 [4, Proposition 1]. *Let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $f \in L_{\alpha\beta}$. Then*

(i) $\lim_{\varphi \to 0} \|\tilde{T}_\varphi f(\bullet)\|_1 = 0$;

(ii) $\int_0^\varepsilon |\tilde{T}_\varphi f(\theta)| \, d\mu_{\alpha\beta}(\varphi) = o(\varepsilon^{2\alpha+2})$, *as* $\varepsilon \to 0$, *for almost all* $\theta \in (0, \pi)$;

(iii) *for* $0 \leqslant \zeta < \min\{\alpha + \beta + 1, 2\beta + 2\}$,

$$\int_0^\pi |\tilde{T}_\varphi f(\theta)| \, d\mu_{\alpha, \beta - \zeta/2}(\varphi) \leqslant M \sin^{-\zeta} \theta \int_0^\pi |f(\varphi)| \, d\mu_{\alpha\beta}(\varphi)$$

2. ESTIMATES OF $\tilde{K}_n^\delta(\varphi)$ WHEN $\delta \geqslant 0$

THEOREM 1. *Let $\alpha \geqslant -\frac{1}{2}$, $\beta \geqslant -\frac{1}{2}$, $\delta \geqslant 0$, and $\delta_0 = [\alpha + \frac{1}{2}] + 1$. Then*

(i) $|\tilde{K}_n^\delta(\varphi)| \leqslant M n^{2\alpha+2}$, $0 \leqslant \varphi \leqslant \pi/2$;

(ii) *if* $0 \leqslant \delta \leqslant \delta_0$,

$$|\tilde{K}_n^\delta(\varphi) - G(\varphi)| \leqslant M n^{\alpha + 1/2 - \delta} \varphi^{-\alpha - 3/2 - \delta} (\pi - \varphi)^{-\beta - 1/2}, \qquad n^{-1} \leqslant \varphi < \pi;$$

(iii) *if* $\delta \geqslant \delta_0$,

$$|\tilde{K}_n^\delta(\varphi) - G(\varphi)| \leqslant M n^{\alpha + 1/2 - \delta_0} \varphi^{-\alpha - 3/2 - \delta_0} (\pi - \varphi)^{-\beta - 1/2}, \qquad n^{-1} \leqslant \varphi < \pi.$$

Throughout the paper, denote by M the constants independent of n, φ and f and dependent of α, β, and δ, possibly different at each occurrence.

To prove Theorem 1, we need the following lemmas. Set $\gamma = \alpha + \beta + 2$ and

$$K_n^{(\alpha, \beta, \delta)}(\varphi) = \frac{1}{A_n^\delta} \sum_{\nu=0}^{\alpha} A_{n-\nu}^\delta \omega_\nu^{(\alpha, \beta)} R_\nu^{(\alpha, \beta)}(\cos \varphi).$$

LEMMA 1.

$$\tilde{K}_n^\delta(\varphi) = \frac{n + \delta + 1}{n + \gamma + \delta + 1} \tilde{K}_n^{\delta+1}(\varphi) + \frac{2\alpha + 2}{n + \gamma + \delta + 1} K_n^{(\alpha+1, \beta+1, \delta)}(\varphi) \sin \varphi.$$

This can be directly calculated from the definition of $\tilde{K}_n^\delta(\varphi)$.

LEMMA 2. *Let* $\alpha \geqslant -\frac{1}{2}$ *and* $\beta \geqslant -\frac{1}{2}$. *Then for* $0 \leqslant \delta \leqslant \alpha + \frac{3}{2}$ *and* $n^{-1} \leqslant \varphi < \pi$,

$$|K_n^{(\alpha, \beta, \delta)}(\varphi)| \leqslant M n^{\alpha + 1/2 - \delta} \varphi^{-\alpha - 3/2 - \delta} (\pi - \varphi)^{-\beta - 1/2}.$$

Lemma 2 is contained in Bonami and Clerc [2, Theorem 2.1].

LEMMA 3. *Let* $\{a_k\}$ *be a sequence and let*

$$E_n^\sigma = \sum_{k=0}^{n} A_{n-k}^\sigma a_k.$$

Then for any complex numbers δ, σ, r *and nonnegative integer* n, *the following equality holds*:

$$\sum_{k=0}^{n} A_{n-k}^\delta r^k a_k = \sum_{j=0}^{n} A_{\sigma+1-j}^j \left[\sum_{v=0}^{n-j} A_{n-j-v}^{\delta-\sigma-1+j} r^v E_v^\sigma \right] (1-r)^j. \qquad (*)$$

Proof. Let $U(\sigma)$ be the right-hand side of $(*)$. Some tedious computation yields $U(\sigma) = U(\sigma+1)$. It is easy to get $U(0) = \sum_{k=0}^{n} A_{n-k}^\delta r^k a_k$. Hence $(*)$ holds for $\sigma = 0, \pm 1, \pm 2, \ldots$. Since $U(\sigma)$ is a polynomial σ, $(*)$ is proved for any complex number σ.

Proof of Theorem 1. Part (i) follows from Szegö [8, (7.32.6)]. In the following, assume that $n^{-1} \leqslant \varphi < \pi$.

We first consider the case when $\delta = \delta_0$. It is clear that $\alpha + \frac{1}{2} < \delta \leqslant \alpha + \frac{3}{2}$. By Lemma 3, we have

$$\begin{aligned} \tilde{K}_n^\delta(\varphi) = 2(\alpha+1) \sin \varphi \int_0^1 \Bigg\{ & K_n^{(\alpha+1, \beta+1, \delta)}(\varphi)\, r^{\alpha+\beta+n+1} + \frac{1}{A_n^\delta} \sum_{j=1}^{\delta+1} A_{\delta+1-j}^j \\ & \times \left[\sum_{v=0}^{n-j} A_{n-j-v}^{j-1} A_v^\delta K_v^{(\alpha+1, \beta+1, \delta)}(\varphi)\, r^{\alpha+\beta+v+1} \right] (1-r)^j \Bigg\}\, dr, \end{aligned}$$

and by (5)

$$G(\varphi) = 2(\alpha+1) \sin \varphi \int_0^1 \left[\sum_{v=0}^{\infty} A_v^\delta K_v^{(\alpha+1, \beta+1, \delta)}(\varphi)\, r^{\alpha+\beta+v+1} \right] (1-r)^{\delta+1}\, dr.$$

It follows that

$$\tilde{K}_n^\delta(\varphi) - G(\varphi) = \Lambda_1 + \Lambda_2 + \Lambda_3, \qquad (7)$$

where

$$\Lambda_1 = 2(\alpha+1)\sin\varphi \left\{ \frac{K_n^{(\alpha+1,\beta+1,\delta)}(\varphi)}{n+\gamma} \right.$$

$$\left. + \frac{1}{A_n^\delta} \sum_{j=1}^{\delta-1} A_{\delta+1-j}^{j} \sum_{v=0}^{n-j} A_{n-j-v}^{j-1} A_v^\delta K_v^{(\alpha+1,\beta+1,\delta)}(\varphi)\, B(v+\gamma, j+1) \right\},$$

$$\Lambda_2 = -2(\alpha+1)\sin\varphi \sum_{v=n-\delta}^{\infty} A_v^\delta K_v^{(\alpha+1,\beta+1,\delta)}(\varphi)\, B(v+\gamma, \delta+2)$$

and

$$\Lambda_3 = \frac{2(\alpha+1)\sin\varphi}{A_n^\delta} \left\{ (\delta+1)\, A_{n-\delta}^\delta K_{n-\delta}^{(\alpha+1,\beta+1,\delta)}(\varphi)\, B(n-\delta+\gamma, \delta+1) \right.$$

$$\left. + \sum_{v=0}^{n-\delta-1} c(n, v, \delta)\, A_v^\delta K_v^{(\alpha+1,\beta+1,\delta)}(\varphi)\, B(v+\gamma, \delta+2) \right\},$$

where

$$c(n, v, \delta) = (v+\gamma+\delta+1)\, A_{n-\delta-v}^{\delta-1} + A_{n-\delta-v-1}^\delta - A_n^\delta$$

and $B(a, b)$ is the beta function.

Since $B(v+\gamma, j) = O((v+1)^{-j})$ and $\alpha+\frac{1}{2} < \delta \leqslant \alpha+\frac{3}{2}$, by Lemma 2 we have that

$$|\Lambda_1| \leqslant M \left\{ n^{\alpha+1/2-\delta} + n^{-\delta} \sum_{j=1}^{\delta-1} \sum_{v=0}^{n-j} (n-j-v+1)^{j-1} (v+1)^{\alpha+1/2-j} \right\}$$

$$\times \varphi^{-\alpha-3/2-\delta}(\pi-\varphi)^{-\beta-1/2}$$

$$\leqslant M n^{\alpha+1/2-\delta} \varphi^{-\alpha-3/2-\delta}(\pi-\varphi)^{-\beta-1/2},$$

$$|\Lambda_2| \leqslant M \sum_{v=n-\delta}^{\infty} v^{\alpha-1/2-\delta} \varphi^{-\alpha-3/2-\delta}(\pi-\varphi)^{-\beta-1/2}$$

$$\leqslant M n^{\alpha+1/2-\delta} \varphi^{-\alpha-3/2-\delta}(\pi-\varphi)^{-\beta-1/2},$$

and

$$|\Lambda_3| \leqslant \frac{M}{n^\delta} \left\{ n^{\alpha+1/2-\delta} + \sum_{v=0}^{n-\delta-1} |c(n, v, \delta)|\, (v+1)^{\alpha-1/2-\delta} \right\}$$

$$\times \varphi^{-\alpha-\delta-3/2}(\pi-\varphi)^{-\beta-1/2}.$$

It is noted that if $n/2 \leqslant v \leqslant n-\delta-1$,

$$|c(n, v, \delta)| \leqslant Mn^{\delta}$$

and if $0 \leqslant v \leqslant n/2$, by Stirling's formula,

$$|c(n, v, \delta)| \leqslant M[n^{\delta-2}(v+1)^2 + n^{\delta-1}].$$

Thus for Λ_3 we have

$$\begin{aligned}|\Lambda_3| \leqslant \frac{M}{n^{\delta}} \Big\{ & n^{\alpha+1/2-\delta} + \sum_{0 \leqslant v \leqslant n/2} [n^{\delta-2}(v+1)^2 + n^{\delta-1}](v+1)^{\alpha-1/2-\delta} \\ & + \sum_{n/2 < v \leqslant n-\delta-1} n^{\delta} v^{\alpha-1/2-\delta} \Big\} \varphi^{-\alpha-\delta-3/2}(\pi-\varphi)^{-\beta-1/2} \\ \leqslant{} & Mn^{\alpha+1/2-\delta} \varphi^{-\alpha-\delta-3/2}(\pi-\varphi)^{-\beta-1/2}.\end{aligned}$$

Substituting the estimates of Λ_1, Λ_2, and Λ_3 into (7) proves (ii) for $\delta = \delta_0$.

Part (iii), when $\delta > \delta_0$, follows by adaptation of the arguments in [2, p. 233].

By Lemma 1,

$$\begin{aligned}&\tilde{K}_n^{\delta}(\varphi) - G(\varphi) \\ &\quad = \frac{2\alpha+2}{n+\gamma+\delta+1} K_n^{(\alpha+1, \beta+1, \delta)}(\varphi) \sin \varphi \\ &\qquad + \frac{n+\delta+1}{n+\gamma+\delta+1} (\tilde{K}_n^{\delta+1}(\varphi) - G(\varphi)) - \frac{\alpha+\beta+2}{n+\gamma+\delta+1} G(\varphi), \qquad (8)\end{aligned}$$

so that by part (iii) and Lemma 2 and Lemma 4 (in Section 3 below), part (ii) for $\delta_0 - 1 \leqslant \delta < \delta_0$ follows. Another application of (8) gives the estimates of $\tilde{K}_n^{\delta}(\varphi)$ for all values of $\delta \geqslant 0$.

3. NORMS OF $\tilde{K}_n^{\delta}(\varphi)$ WHEN $0 \leqslant \delta \leqslant \alpha + \frac{1}{2}$

LEMMA 4. *For* $0 < \varphi < \pi$,

$$G(\varphi) = \frac{\Gamma(\alpha+3/2)}{2^{\alpha+\beta+1}\Gamma(1/2)\,\Gamma(\alpha+1)} \sin^{-2\alpha-2}\varphi/2 \cos\varphi/2 + O(\varphi^{-2\alpha-1}(\pi-\varphi)).$$

Proof. First we have

$$P^{(\alpha+1, \beta+1)}(s, \varphi) = \frac{\Gamma(\alpha+\beta+4)(1-s)\,\Lambda^{-\alpha-5/2}(s, \varphi)}{2^{\alpha+\beta+3}\Gamma(\alpha+2)\,\Gamma(\beta+2)(1+s)^{\beta-\alpha-1}} F\left[\frac{4s\cos^2\varphi/2}{(1+s)^2}\right],$$

where $\Lambda(s,\varphi)=1-2s\cos\varphi+s^2$, $F[x]=F[(\beta-\alpha)/2,\ (\beta-\alpha-1)/2;\ \beta+2; x]$ and $F[a,b;c;x]$ is the Gauss hypergeometric function (see [3] or [4, (5.3)]. Since $F[a,b;c;x]$ is a continuous function of x for $0\leqslant x\leqslant 1$ when $\operatorname{Re}(c-a-b)>0$ and $c\neq 0,\ -1,\ -2,\ \ldots$, it follows from [3, 2–1(7)]

$$\frac{d}{dx}F[a,b;c;x]=\frac{ab}{c}F[a+1,b+1;c+1;x]$$

that

$$\left|F\left[\frac{4s\cos^2\varphi/2}{(1+s)^2}\right]-F[1]\right|\leqslant M\Lambda(s,\varphi),$$

and then

$$P^{(\alpha+1,\beta+1)}(s,\varphi)=A_{\alpha\beta}\frac{(1-s)\,\Lambda^{-\alpha-5/2}(s,\varphi)}{(1+s)^{\beta-\alpha-1}}\{1+O(\Lambda(s,\varphi))\},$$

where

$$\begin{aligned}A_{\alpha\beta}&=\frac{\Gamma(\alpha+\beta+4)}{2^{\alpha+\beta+3}\Gamma(\alpha+2)\,\Gamma(\beta+2)}F\left[\frac{\beta-\alpha}{2},\frac{\beta-\alpha-1}{2};\beta+2;1\right]\\&=\frac{\Gamma(\alpha+5/2)}{\Gamma(1/2)\,\Gamma(\alpha+2)}\qquad\text{by [3, 2–8(46)].}\end{aligned}$$

Thus by (5)

$$\begin{aligned}G(\varphi)&=2(\alpha+1)\,A_{\alpha\beta}\int_{1/2}^{1}\frac{s^{\alpha+\beta+1}(1-s)}{(1+s)^{\beta-\alpha-1}}\Lambda^{-\alpha-5/2}(s,\varphi)\sin\varphi\,ds\\&\quad+O(\varphi^{-2\alpha}(\pi-\varphi))\\&=\frac{(\alpha+1)\,A_{\alpha\beta}}{2^{\beta-\alpha-2}}\int_{1/2}^{1}\frac{(1-s)\sin\varphi}{\Lambda^{\alpha+5/2}(s,\varphi)}ds+O(\varphi^{-2\alpha-1}(\pi-\varphi)),\end{aligned}$$

since $s^{\alpha+\beta+1}(1+s)^{\alpha-\beta+1}=2^{\alpha-\beta+1}(1+O(1-s))$ for $\frac{1}{2}<s<1$.

It is easy to get, for $\frac{1}{2}<s<1$,

$$\frac{1}{\Lambda^{\alpha+5/2}(s,\varphi)}=\frac{1}{[(1-s)^2+2(1-\cos\varphi)]^{\alpha+5/2}}+O\left(\frac{(1-s)\,\varphi^2}{\Lambda^{\alpha+7/2}(s,\varphi)}\right).$$

Therefore,

$$G(\varphi)=\frac{(\alpha+1)\,A_{\alpha\beta}}{2^{\beta-\alpha-2}}\int_{1/2}^{1}\frac{(1-s)\sin\varphi}{[(1-s)^2+2(1-\cos\varphi)]^{\alpha+5/2}}\,ds+O(\varphi^{-2\alpha-1}(\pi-\varphi))$$
$$=\frac{\Gamma(\alpha+3/2)}{2^{\alpha+\beta+1}\Gamma(1/2)\,\Gamma(\alpha+1)}\sin^{-2\alpha-2}\varphi/2\cos\varphi/2+O(\varphi^{-2\alpha-1}(\pi-\varphi)).$$

Lemma 5. *Let $\alpha\geqslant-\frac{1}{2}$ and $\beta\geqslant-\frac{1}{2}$. Then*

$$\frac{2\alpha+2}{n+\gamma+\delta+1}\int_0^\pi |K_n^{(\alpha+1,\,\beta+1,\,\delta)}(\varphi)|\sin\varphi\,d\mu_{\alpha\beta}(\varphi)$$
$$=\frac{2^{3/2-\alpha}\Gamma(\alpha+\frac{3}{2})}{\pi^{3/2}\Gamma(\alpha+1)}\log n+O(1),\qquad \delta=\alpha+\tfrac{1}{2};$$
$$=B_{\alpha\beta\delta}n^{\alpha+1/2-\delta}+O(n^{\alpha-\beta-\delta-1})+O(n^{\alpha-1/2-\delta})+O(1),$$
$$0\leqslant\delta<\alpha+\tfrac{1}{2},$$

where

$$B_{\alpha\beta\delta}=\frac{\Gamma(\delta+1)\,\Gamma((2\alpha-2\delta+1)/4)\,\Gamma((2\beta+3)/4)}{2^{\delta-1}\pi^{3/2}\Gamma(\alpha+1)\,\Gamma((\alpha+\beta-\delta+2)/2)}.$$

If $\beta=-\frac{1}{2}$ or $\delta=\alpha-\frac{1}{2}$, then $O(n^{\alpha-\beta-\delta-1})+O(n^{\alpha-1/2-\delta})$ in the second case must be replaced by $O(n^{\alpha-\delta-1/2}\log n)$.

Both of the two cases in Lemma 5 can be proved by the same manner in [6, Section 3].

Theorem 2. *Let $\alpha\geqslant-\frac{1}{2}$ and $\beta\geqslant-\frac{1}{2}$. Then*

$$\tilde{L}_n^\delta=\int_0^\pi|\tilde{K}_n^\delta(\varphi)|\,d\mu_{\alpha\beta}(\varphi)$$
$$=\frac{2\Gamma(\alpha+\frac{3}{2})}{\Gamma(1/2)\,\Gamma(\alpha+1)}\log n+O(1),\qquad \delta=\alpha+\tfrac{1}{2};$$
$$=B_{\alpha\beta\delta}n^{\alpha+1/2-\delta}+O(n^{\alpha-\beta-\delta-1})+O(n^{\alpha-1/2-\delta})+O(\log n)+O(1),$$
$$0\leqslant\delta<\alpha+\tfrac{1}{2},$$

where $B_{\alpha\beta\delta}$ is given in Lemma 5. If $\beta=-\frac{1}{2}$ or $\delta=\alpha-\frac{1}{2}$, then $O(n^{\alpha-\beta-\delta-1})+O(n^{\alpha-1/2-\delta})$ in the second case must be replaced by $O(n^{\alpha-\delta-1/2}\log n)$.

It is clear that when $\alpha=-\frac{1}{2}$ and $\delta=0$, $\tilde{L}_n^0=(2/\pi)\log n+O(1)$ which coincides with that in the classical case (cf. [9, p. 67, (12.3)]).

Proof. By (8) and Theorem 1(i),

$$\tilde{L}_n^\delta = \int_{1/n}^{\pi} |\tilde{K}_n^\delta(\varphi) - G(\varphi) + G(\varphi)| \, d\mu_{\alpha\beta}(\varphi) + O(1)$$

$$= \int_{1/n}^{\pi} \left| \frac{2\alpha+2}{n+\gamma+\delta+1} K_n^{(\alpha+1, \beta+1, \delta)}(\varphi) \sin\varphi + G(\varphi) \right| d\mu_{\alpha\beta}(\varphi)$$

$$+ O(1) \int_{1/n}^{\pi} [|\tilde{K}_n^{\delta+1}(\varphi) - G(\varphi)| + n^{-1} G(\varphi)] \, d\mu_{\alpha\beta}(\varphi) + O(1).$$

If $\alpha - \frac{1}{2} < \delta \leqslant \alpha + \frac{1}{2}$, it follows from Theorem 1(ii) and Lemma 4, the second term on the right-hand side above is $O(1)$, so that

$$\tilde{L}_n^\delta = \int_{1/n}^{\pi} \left| \frac{2\alpha+2}{n+\gamma+\delta+1} K_n^{(\alpha+1, \beta+1, \delta)}(\varphi) \sin\varphi + G(\varphi) \right| d\mu_{\alpha\beta}(\varphi) + O(1). \tag{9}$$

By Lemma 3

$$\int_{1/2}^{\pi} |G(\varphi)| \, d\mu_{\alpha\beta}(\varphi) = \frac{2\Gamma(\alpha+3/2)}{\Gamma(1/2)\,\Gamma(\alpha+1)} \log n + O(1).$$

Applying this and Lemma 5 to (9) proves the theorem for $\alpha - \frac{1}{2} < \delta < \alpha + \frac{1}{2}$.

To evaluate $\tilde{L}_n^{\alpha+1/2}$, we note that

$$K_n^{(\alpha+1, \beta+1, \alpha+1/2)}(\varphi) = c(n)\, R_n^{(2\alpha+5/2, \beta+1)}(\cos\varphi) + \sum_{j=1}^{\infty} c_j(n)\, K_n^{(\alpha+1, \beta+1, \alpha+j+1/2)}(\varphi)$$

by [8, (9.41.13)] or [6, (2.2)], where

$$c(n) = \frac{2^{-2\alpha-\beta-7/2}\Gamma(\alpha+3/2)}{\Gamma(\alpha+2)\,\Gamma(2\alpha+7/2)} n^{2\alpha+4} + O(n^{2\alpha+3}) \tag{10}$$

and $|c_j(n)| \leqslant Mj^{-2\alpha-\beta-13/2}$. By the argument similar to that in [6, Section 3], it is not difficult to find that the contribution from the second term above to the integral in (9) is $O(1)$ so that

$$\tilde{L}_n^{\alpha+1/2} = \int_{1/n}^{\pi} \left| \frac{(2\alpha+2)\, c(n)}{n+\gamma+\delta+1} \times R_n^{(2\alpha+5/2, \beta+1)}(\cos\varphi) \sin\varphi + G(\varphi) \right| d\mu_{\alpha\beta}(\varphi) + O(1).$$

By (10), Lemma 4, and [8, (4.1.3), (7.32.6)], the contribution of the interval $\pi/2 \leqslant \varphi < \pi$ to the integral above is $O(1)$. Hence, by (10) and Lemma 4 again, using [8, (8.21.18)],

$$R_n^{(2\alpha+5/2,\,\beta+1)}(\cos\varphi) = \frac{n^{-1/2}\Gamma(2\alpha+7/2)\,\Gamma(n+1)}{\Gamma(1/2)\,\Gamma(n+2\alpha+7/2)} \sin^{-2\alpha-3}\varphi/2\,\cos^{-\beta-3/2}\varphi/2$$
$$\times\{\cos(N\varphi+\eta)+O(n^{-1}\varphi^{-1})\}, \qquad 1/n \leqslant \varphi \leqslant \pi/2,$$

yields

$$\begin{aligned}\tilde{L}_n^{\alpha+1/2} &= \frac{E(\alpha)}{2^{\alpha+\beta+2}} \int_{1/n}^{\pi/2} \left|\frac{\cos^{-\beta-1/2}\varphi/2}{2^{\alpha+1/2}}\cos(N\varphi+\eta)+\cos\varphi/2\right| \\ &\quad\times \sin^{-2\alpha-2}\varphi/2\,d\mu_{\alpha\beta}(\varphi)+O(1) \\ &= \frac{E(\alpha)}{2^{\alpha+\beta+2}} \int_{1/n}^{\pi/2} \left|\frac{\cos(N\varphi+\eta)}{2^{\alpha+1/2}}+1\right| \sin^{-2\alpha-2}\varphi/2\,d\mu_{\alpha\beta}(\varphi)+O(1) \\ &= \frac{E(\alpha)}{2^{\alpha+\beta+2}} \int_{1/n}^{\pi/2} \sin^{-2\alpha-2}\varphi/2\,d\mu_{\alpha\beta}(\varphi)+O(1) \\ &= E(\alpha)\log n+O(1),\end{aligned}$$

where $E(\alpha)=2\Gamma(\alpha+3/2)/\Gamma(1/2)\,\Gamma(\alpha+1)$, $N=n+(2\alpha+\beta+9/2)/2$, $\eta=-(\alpha+3/2)\pi$.

Another application of Lemma 1 and Lemma 5 gives the desired result for all values of $0 \leqslant \delta < \alpha+\frac{1}{2}$.

4. CONVERGENCE OF $\tilde{S}_n^\delta(f;\theta)$ FOR $\delta \geqslant \alpha+\frac{1}{2}$

4.1. *The Convergence of $\tilde{S}_n^\delta(f,\theta)$ when $\delta > \alpha+\frac{1}{2}$*

THEOREM 3. *Let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and $\delta > \alpha+\frac{1}{2}$ and let $f \in L_{\alpha\beta}$. If for some $\theta \in (0,\pi)$,*

$$\int_0^\varepsilon |\tilde{T}_\varphi f(\theta)|\,d\mu_{\alpha\beta}(\varphi) = o(\varepsilon^{2\alpha+2}) \qquad as \quad \varepsilon \to 0, \tag{11}$$

then

$$\lim_{n\to\infty}\{\tilde{S}_n^\delta(f;\theta)-\tilde{f}_{1/n}(\theta)\}=0, \tag{12}$$

so that $\lim_{n\to\infty}\tilde{S}_n^\delta(f;\theta)=\tilde{f}(\theta)$ almost everywhere.

Proof. Without loss of generality, assume that $\alpha > -\frac{1}{2}$, $\beta \geqslant -\frac{1}{2}$, and $\alpha + \frac{1}{2} < \delta \leqslant \delta_0 = [\alpha + \frac{1}{2}] + 1$.

Write $\tilde{S}_n^\delta(f; \theta) - \tilde{f}_{1/n}(\theta) = U + V + W$, where

$$U = \int_0^{1/n} \tilde{T}_\varphi f(\theta) \cdot \tilde{K}_n^\delta(\varphi)\, d\mu_{\alpha\beta}(\varphi),$$

$$V = \int_{1/n}^{\pi/2} \tilde{T}_\varphi f(\theta) \cdot (\tilde{K}_n^\delta(\varphi) - G(\varphi))\, d\mu_{\alpha\beta}(\varphi),$$

$$W = \int_{\pi/2}^{\pi} \tilde{T}_\varphi f(\theta) \cdot (\tilde{K}_n^\delta(\varphi) - G(\varphi))\, d\mu_{\alpha\beta}(\varphi).$$

First by Theorem 1(i) and (11), we have

$$|U| \leqslant Mn^{2\alpha+2} \int_0^{1/n} |\tilde{T}_\varphi f(\theta)|\, d\mu_{\alpha\beta}(\varphi) = o(1) \qquad \text{as} \quad n \to \infty.$$

For V, by Theorem 1(ii) and (11), it follows from a standard argument that

$$|V| \leqslant Mn^{\alpha+1/2-\delta} \int_{1/n}^{\pi/2} |\tilde{T}_\varphi f(\theta)|\, \varphi^{-\alpha-3/2-\delta}\, d\mu_{\alpha\beta}(\varphi) = o(1) \qquad \text{as} \quad n \to \infty.$$

At last by Theorem 1(iii) and Proposition 1(iii), we have

$$\begin{aligned} |W| &\leqslant Mn^{\alpha+1/2-\delta} \int_{\pi/2}^{\pi} |\tilde{T}_\varphi f(\theta)|\, (\pi - \varphi)^{-\beta-1/2}\, d\mu_{\alpha\beta}(\varphi) \\ &\leqslant Mn^{\alpha+1/2-\delta} \sin^{-\beta-1/2}\theta \int_0^{\pi} |f(\varphi)|\, d\mu_{\alpha\beta}(\varphi) \\ &= o(1) \qquad \text{as} \quad n \to \infty. \end{aligned}$$

This proves (12), and by Proposition 1(ii) we finish the proof.

4.2. *The Convergence of $\tilde{S}_n^{\alpha+1/2}(f; \theta)$ at the Critical Index*

THEOREM 4 (Equiconvergence theorem). *Let $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and let $f \in L_{\alpha\beta}$. If for some $\theta \in (0, \pi)$, (11) is satisfied, then the following statements are equivalent*:

(i) $\lim_{n\to\infty} \{\tilde{S}_n^{\alpha+1/2}(f;\theta) - \tilde{f}_{1/n}(\theta)\} = 0;$

(ii) $\lim_{n\to\infty} \frac{1}{n} \int_0^{\pi} \tilde{T}_\varphi f(\theta) \cdot K_n^{(\alpha+1, \beta+1, \alpha+1/2)}(\varphi) \sin\varphi \, d\mu_{\alpha\beta}(\varphi) = 0;$

(iii) $\lim_{n\to\infty} n^{2\alpha+3} \int_0^{\pi/2} \tilde{T}_\varphi f(\theta) \cdot R_n^{(2\alpha+5/2, \beta+1)}(\varphi) \sin\varphi \, d\mu_{\alpha\beta}(\varphi) = 0;$

(iv) $\lim_{n\to\infty} n^{1/2} \int_{1/n}^{\pi/2} \tilde{T}_\varphi f(\theta) \cdot J_{2\alpha+5/2}(N\varphi) \, \varphi^{1/2} \, d\mu_{-1, \beta/2-1/4}(\varphi) = 0;$

(v) $\lim_{n\to\infty} \int_{1/n}^{\pi/2} \tilde{T}_\varphi f(\theta) \cdot \sin(N\varphi - \alpha\pi) \, d\mu_{-1, \beta/2-1/4}(\varphi) = 0,$

where $J_{2\alpha+5/2}(t)$ *is the Bessel function of the first kind of order* $2\alpha + \frac{5}{2}$ *and* $N = n + (2\alpha + \beta + 9/2)/2$.

Proof. By Lemma 1. We have

$$\begin{aligned} &\tilde{S}_n^{\alpha+1/2}(f;\theta) - \tilde{f}_{1/n}(\theta) \\ &\quad = \frac{2\alpha+2}{n+\gamma+\delta+1} \int_0^{\pi} \tilde{T}_\varphi f(\theta) \cdot K_n^{(\alpha+1, \beta+1, \alpha+1/2)}(\varphi) \sin\varphi \, d\mu_{\alpha\beta}(\varphi) \\ &\qquad + \frac{n+\delta+1}{n+\gamma+\delta+1} (\tilde{S}_n^{\alpha+3/2}(f;\theta) - \tilde{f}_{1/n}(\theta)) - \frac{\alpha+\beta+2}{n+\gamma+\delta+1} \tilde{f}_{1/n}(\theta). \end{aligned} \tag{13}$$

By Theorem 3, the second term on the right-hand side of (13) is $o(1)$ if (11) is satisfied at $\theta \in (0, \pi)$. In the meantime, under the same condition it is easy to find that the last term is $o(1)$ as $n \to \infty$ by use of Lemma 4. This proves the equivalence of (i) and (ii). The equivalence of (ii) with (iii)–(v) can be proved by the same manner as in Li [6, Section 5].

Just as the case of ordinary Jacobi series (cf. Li [6]), the equiconvergence forms of $\tilde{S}_n^{\alpha+1/2}(f;\theta)$ in Theorem 4 can be used to deduce some convergence criteria for $\tilde{S}_n^{\alpha+1/2}(f;\theta)$ such as those due to Lebesgue, Salem, and Young. Here we only state the theorem of Lebesgue type, the proof of which is completely the same as that of [6, Theorem 6.1].

THEOREM 6. *Let* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ *and let* $f \in L_{\alpha\beta}$. *If for some* $\theta \in (0, \pi)$, (11) *is satisfied, in addition,*

$$\lim_{n\to\infty} \int_{\varepsilon}^{\pi/2} \frac{|\tilde{T}_{\varphi+\varepsilon} f(\theta) - \tilde{T}_\varphi f(\theta)|}{\varphi} \, d\varphi = 0, \tag{14}$$

then

$$\lim_{n \to \infty} \{ \tilde{S}_n^{\alpha+1/2}(f; \theta) - \tilde{f}_{1/n}(\theta) \} = 0.$$

Moreover, if (14) *holds for a.e.* $\theta \in (0, \pi)$, *then* $\lim_{n \to \infty} \tilde{S}_n^{\alpha+1/2}(f; \theta) = \tilde{f}(\theta)$ *for a.e.* $\theta \in (0, \pi)$.

REFERENCES

1. H. Bavinck, "Approximation Processes for Fourier–Jacobi Expansions," Math. Centrum, Amsterdam, 1971.
2. A. Bonami and J. L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmonique sphérique, *Trans. Amer. Math. Soc.* **183** (1973), 223–263.
3. A. Erdélyi *et al.*, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.
4. Zh.-K. Li, Conjugate Jacobi series and conjugate functions, *J. Approx. Theory* **86** (1996), 179–196.
5. Zh.-K. Li, Hardy spaces for Jacobi expansions I. The basic theory, *Analysis* **16** (1996), 27–49.
6. Zh.-K. Li, Pointwise convergence of Fourier–Jacobi series, *Approx. Theory Appl. (N.S.)* **11**, No. 4 (1995), 58–77.
7. B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–92.
8. G. Szegö, "Orthogonal Polynomials," Colloq. Publ., Vol. 23, 4th ed., Amer. Math. Soc., Providence, RI, 1975.
9. A. Zygmund, "Trigonometric Series," Vol. I, 2nd ed., Cambridge Univ. Press, London/New York, 1959.